Invariant trace of flat space chiral higher-spin algebra as scattering amplitudes

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Introduction

Motivation

Higher-spin gauge theories are theories involving massless spin-s fields with s>2

Massless fields of spin > 1/2 are all gauge fields. The larger the spin the larger is the associated symmetry

Rich symmetries are good in physics and mathematics (e.g. improve quantum behaviour, quantum gravity?)

Current status

<u>In flat space:</u>

No-go theorems, which, for some assumptions, rule out non-trivial interactions of HS gauge fields.

[Weinberg ’64; Coleman, Mandula ’67]

There exist a chiral (self-dual) higher-spin theory in 4d. Scattering in this theory is to large extent trivial as a consequence of self-duality

[Metsaev ’91; DP, Skvortsov ’16]

Current status

<u>In AdS space:</u>

There are various construction, most notably, the Vasiliev theory.

[Vasiliev '91; Vasiliev '2003]

Existence supported by the holographic duality with the vector models (CFT's)

[Sezgin, Sundell '02; Klebanov, Polyakov '02]

Ongoing debates concerning locality in these theories - theories are non-local in the conventional sense with rather exotic amplitudes

Current status

Other setups:

Higher-spin theories in 3d can be constructed as the Chern-Simons theories

[Blencowe '89]

Conformal HS theories can be constructed rather explicitly

[Tseytlin '02; Segal '02]

This talk

In the present talk

Rich higher-spin *symmetries alone is a powerful tool to construct HS theories*. Below we will discuss how higher-spin theories in flat space can be constructed by requiring proper symmetries of the S-matrix. Thus, we aim to go beyond the self-dual sector (chiral theories) and have more non-trivial scattering.

Constraints on the S-matrix from symmetries

How this usually works

Poincare global symmetry:

1) fixes 3-pt amplitudes completely
2) fixes 4-pt amplitudes up to a function of two Mandelstam variables
3) higher-point functions – more independent Mandelstam variables

Global conformal symmetry

1) fixes 3-pt correlators completely
2) fixes 4-pt correlators up to a function of two conformally invariant cross-ratios
3) higher-point functions – more independent conformally invariant cross-ratios

More symmetry – more constraints, e.g. supersymmetry, Yangian symmetry, etc – further reduce possibilities for consistent amplitudes

Higher-spin symmetric S-matrices

Higher-spin symmetries are so rich that:

They either fix the S-matrix almost uniquely (up to an overall factor for n-point amplitude)

Or rule out non-trivial S-matrices completely

So, non-trivial higher-spin theories appear exactly on the border-line: these are very symmetric, but if we ask a bit too much, interactions are ruled out completely.

Higher-spin gauge theories

These expectations are based on, in particular,

In flat space, there is a number of no-go theorems, that rule out non-trivial scattering of HS gauge fields.

[Coleman, Mandula '67]

In the AdS space, higher-spin theories have a holographic description as simple vector models. When higher-spin symmetry is unbroken, the CFT correlators are fixed (almost) uniquely. Higher-spin symmetry alone fixes n-point correlators up to an overall factor.

[Maldacena, Zhiboedov '11]

May be in flat space we ask a bit too much and if the assumptions of the no-go theorems are slightly relaxed, we may get non-trivial higher-spin S-matrices?

This talk

Try to carry the procedure that allows us to fix the S-matrix of higher-spin gauge fields in the AdS space over to flat space. Particular form of the AdS space procedure that we will follow:

[Colombo, Sundell '12; Didenko, Skvortsov '12; Gelfond, Vasiliev '13]

Plan

1) Generalities on the S-matrix for higher-spin gauge fields

2) 4d case, sl(2,C) spinors and the spinor-helicity formalism

3) Construction of higher-spin invariant amplitudes in AdS as invariant traces of the higher-spin algebra

4) Extension to flat space

S-matrices for higher-spin gauge fields: general discussion

Summary of requirements

1) *Spectrum*. Each spin-s field comes with the global symmetry parameter. In the covariant approach one can show that it takes values in rank-(s-1) symmetric traceless Killing tensors.

2) *Jacobi identity*. There should exist a Lie algebra with this spectrum. This algebra should have Poincare subalgebra, under which all generators decompose into Killing tensors.

3) *Fields = its representations*. There should exist an on-shell field representation of this Lie algebra. With respect to the Poincare subalgebra this representation should split into massless higher-spin fields.

4) The S-matrix should be invariant under transformations of the external lines in the on-shell field representation

These conditions are very hard to satisfy!

The AdS case

Conceptually everything remains the same – partial derivatives just need to be replaced with the AdS background covariant derivatives

sl(2,C) spinors and the spinor-helicity formalism

SL(2,C) spinors

Four dimensional Lorentz algebra is isomorphic to

$$so(3,1) \sim sl(2,\mathbb{C}).$$

Accordingly Lorentz vectors can be converted to sl(2,C) bispinors and back

$$p_{\alpha\dot{\alpha}} \equiv p_a(\sigma^a)_{\alpha\dot{\alpha}}, \qquad p_a = -\frac{1}{2}(\sigma_a)^{\dot{\alpha}\alpha}p_{\alpha\dot{\alpha}}.$$

Here sigma are the Pauli matrices. For light-like vectors (massless momenta) one has

$$p^a p_a = 0 \qquad \Leftrightarrow \qquad \det(p_{\alpha\dot{\alpha}}) = 0 \qquad \Leftrightarrow \qquad p_{\alpha\dot{\alpha}} = -\lambda_\alpha \bar{\lambda}_{\dot{\alpha}}.$$

For real positive energy momenta

$$\bar{\lambda}_{\dot{\alpha}} = (\lambda_\alpha)^*$$

We will relax this condition: lambda's are independent, hence, momenta are complex.

Polarisation vectors

In the spinor-helicity formalism one uses a specific representation for polarisation vectors. Helicity +1 and -1 polarisation vectors for spin-1 field are given by

$$\varepsilon_a^+ = \frac{1}{\sqrt{2}} \frac{(\sigma_a)^{\alpha\dot{\alpha}} \bar{\lambda}_{\dot{\alpha}} \mu_\alpha}{\mu^\beta \lambda_\beta}, \qquad \varepsilon_a^- = (\varepsilon_a^+)^*$$

Here mu is the auxiliary 'reference' spinor. Changes in mu = gauge transformations.

Using these polarisation vectors and lambda's instead of momenta in the Feynman rules, we get something like

$$M^{+1,+1,-1} = \frac{[12]^4}{[12][23][31]},$$

where

$$[ij] \equiv \bar{\lambda}^i_{\dot{\alpha}} \bar{\lambda}^j_{\dot{\beta}} \epsilon^{\dot{\alpha}\dot{\beta}}, \qquad \langle ij \rangle \equiv \lambda^i_\alpha \lambda^j_\beta \epsilon^{\alpha\beta}$$

Massless on-shell fields

In terms of sl(2,C) spinors massless representations are realised by

$$J_{\alpha\beta} = i\left(\lambda_\alpha \frac{\partial}{\partial\lambda^\beta} + \lambda_\beta \frac{\partial}{\partial\lambda^\alpha}\right),$$

$$\bar{J}_{\alpha\beta} = i\left(\bar{\lambda}_{\dot\alpha} \frac{\partial}{\partial\bar{\lambda}^{\dot\beta}} + \bar{\lambda}_{\dot\beta} \frac{\partial}{\partial\bar{\lambda}^{\dot\alpha}}\right),$$

$$P_{\alpha\dot\alpha} = -\lambda_\alpha \bar{\lambda}_{\dot\alpha},$$

which act on functions $\Phi(\lambda, \bar{\lambda})$ on $\mathbb{C}^2/\{0\}$. One can introduce the helicity operator

$$H \equiv \frac{1}{2}\left(\bar{N} - N\right), \qquad \bar{N} \equiv \bar{\lambda}^{\dot\alpha} \frac{\partial}{\partial\bar{\lambda}^{\dot\alpha}}, \qquad N \equiv \lambda^\alpha \frac{\partial}{\partial\lambda^\alpha}$$

Its eigenspaces

$$H\Phi_h = h\Phi_h$$

are irreducible helicity h massless representations. Spin s = helicitiy +s and helicity -s. For bosonic fields

$$h \in \mathbb{Z}, \qquad \Phi(-\lambda, -\bar{\lambda}) = \Phi(\lambda, \bar{\lambda})$$

Practical convenience

Instead of a multiplet of fields $\varphi^{a(s)}(p)$ with trace, divergence, on-shell constraints and gauge invariance, now we have a single field $\Phi(\lambda, \bar{\lambda})$.

Massless on-shell fields in AdS

Massless fields in AdS can be realised as

$$J_{\alpha\beta} = i\left(\lambda_\alpha \frac{\partial}{\partial\lambda^\beta} + \lambda_\beta \frac{\partial}{\partial\lambda^\alpha}\right),$$

$$\bar{J}_{\alpha\beta} = i\left(\bar{\lambda}_{\dot\alpha} \frac{\partial}{\partial\bar{\lambda}^{\dot\beta}} + \bar{\lambda}_{\dot\beta} \frac{\partial}{\partial\bar{\lambda}^{\dot\alpha}}\right),$$

$$P_{\alpha\dot\alpha} = -\lambda_\alpha\bar{\lambda}_{\dot\alpha} + \frac{\partial}{\partial\lambda^\alpha}\frac{\partial}{\partial\bar{\lambda}^{\dot\alpha}}.$$

The rest remains the same except that helicity +s and helicity -s are equivalent representations.

Higher-spin invariant amplitudes in AdS

[Colombo, Sundell ’12; Didenko, Skvortsov ’12; Gelfond, Vasiliev ’13]

Higher-spin algebra

Higher-spin algebra in AdS space is defined in terms of the *associative* star product

$$(\Psi_1 \star \Psi_2)(\lambda_3, \bar{\lambda}_3) \equiv \int d^2\lambda_1 d^2\bar{\lambda}_1 d^2\lambda_2 d^2\bar{\lambda}_2 \Psi_1(\lambda_1, \bar{\lambda}_1)\Psi_2(\lambda_2, \bar{\lambda}_2) e^{i([21]+[13]+[32])} e^{i(\langle 21\rangle+\langle 13\rangle+\langle 32\rangle)}.$$

The Lie algebra commutator is just

$$[\Psi_1, \Psi_2]_\star = \Psi_1 \star \Psi_2 - \Psi_2 \star \Psi_1.$$

The AdS isometries so(3,2) are generated by commutators with quadratic polynomials

$$P_{\alpha\dot{\alpha}} \sim \lambda_\alpha \bar{\lambda}_{\dot{\alpha}}, \qquad J_{\alpha\alpha} \sim \lambda_\alpha \lambda_\alpha, \qquad \bar{J}_{\dot{\alpha}\dot{\alpha}} \sim \bar{\lambda}_{\dot{\alpha}} \bar{\lambda}_{\dot{\alpha}}.$$

The space of Psi under so(3,2) then splits into the direct sum of traceless Killing tensors

[Fradkin, Vasiliev ‘87]

On-shell fields

The representation of this algebra, which carries on-shell fields is constructed as

$$\delta_\Psi \Phi = -\Psi \star \Phi + \Phi \star \tilde{\Psi}, \qquad \tilde{\Psi}(\lambda, \bar{\lambda}) \equiv \Psi(-\lambda, \bar{\lambda}) = \Psi(\lambda, -\bar{\lambda}).$$

It can then be checked that for Psi that correspond to the so(3,2) generators, Phi, indeed, transform as massless on-shell fields.

Invariants of the higher-spin algebra

The star product features a trace, which is *cyclic* for bosonic fields

$$\mathrm{Tr}(\Psi_1 \star \Psi_2) = \mathrm{Tr}(\Psi_2 \star \Psi_1), \qquad \mathrm{tr}(\Psi) \equiv \int d^2\lambda d^2\bar{\lambda}\Psi(\lambda,\bar{\lambda})\delta^2(\bar{\lambda})\delta^2(\lambda) = \Psi(0,0).$$

Together with associativity, this implies that

$$G_n \equiv \mathrm{tr}(\Psi_1 \star \Psi_2 \star \cdots \star \Psi_n)$$

Is invariant under higher-spin algebra transformations

$$\delta_\xi \Psi = [\Psi, \xi]_\star .$$

Thus one constructs invariants of the higher-spin algebra. Here, however, Psi transform as Killing tensors, not as fields.

Invariant scattering amplitudes

One can show that if

$$\delta_\xi \Phi = -\xi \star \Phi + \Phi \star \tilde{\xi}$$

then $\Psi = \Phi \star \delta^2(\lambda)$ transforms as $\delta_\xi \Psi = [\Psi, \xi]_\star$.

[Didenko, Vasiliev '09]

Accordingly,

$$G_n \equiv \mathrm{tr}(\Phi_1 \star \delta^2(\lambda) \star \Phi_2 \star \delta^2(\lambda) \star \cdots \star \ \Phi_n \star \delta^2(\lambda)),$$

where Phi's now transform as on-shell fields is HS-invariant.

These give candidate higher-spin amplitudes, which have been *checked holographically*.

Invariant scattering amplitudes

Amplitude

$$G_n \equiv \mathrm{tr}(\Phi_1 \star \delta^2(\lambda) \star \Phi_2 \star \delta^2(\lambda) \star \cdots \star \ \Phi_n \star \delta^2(\lambda)),$$

is superficially chiral (delta-functions on lambda but not on lambda bar).

One can show that

$$\bar{G}_n \equiv \mathrm{tr}(\Phi_1 \star \delta^2(\bar{\lambda}) \star \Phi_2 \star \delta^2(\bar{\lambda}) \star \cdots \star \ \Phi_n \star \delta^2(\bar{\lambda})),$$

is invariant with respect to higher-spin symmetries as well. By adding these, we obtain a parity-invariant amplitude

Invariant scattering amplitudes

More explicitly, for 3-point functions one finds

$$G_3 = \int d^2\lambda_1 d^2\bar{\lambda}_1 d^2\lambda_2 d^2\bar{\lambda}_2 d^2\lambda_3 d^2\bar{\lambda}_3 \Phi_1(\lambda_1,\bar{\lambda}_1)\Phi_2(\lambda_2,\bar{\lambda}_2)\Phi_3(\lambda_3,\bar{\lambda}_3)$$
$$e^{i[12]}\delta^2(\bar{\lambda}_1+\bar{\lambda}_2+\bar{\lambda}_3)e^{i(\langle 21\rangle+\langle 13\rangle+\langle 32\rangle)}.$$

The kernel of this integral can be regarded as an amplitude

$$A_3 = e^{i[12]}\delta^2(\bar{\lambda}_1+\bar{\lambda}_2+\bar{\lambda}_3)e^{i(\langle 21\rangle+\langle 13\rangle+\langle 32\rangle)}.$$

Extension to flat space

Chiral higher-spin theory

In 4d Minkowski flat space there exists the so-called *chiral higher-spin theory*. It is constructed in the light-cone gauge, by requiring Poincare invariance of the action. It has all integer helicities.

[Matsaev '91; DP, Skvortsov '16]

In a well-defined sense it can be regarded as the *higher-spin generalisation* of *self-dual Yang-Mills* theory and *self-dual gravity*. It is also chiral, the action is not real in the (3,1) signature.

[DP '17]

Other properties carry over from self-dual theories: integrability, vanishing of tree-level n-point amplitudes with n>3. Three-point amplitude is

$$M_3^{h_1,h_2,h_3} = g\frac{\ell^{h-1}}{(h-1)!}[12]^{h_1+h_2-h_3}[23]^{h_2+h_3-h_1}[31]^{h_3+h_1-h_2}, \qquad h \equiv h_1+h_2+h_3.$$

To be non-trivial require complex momenta (feature of massless 3-pt amplitudes)

Chiral higher-spin theory

Chiral higher-spin theories have also been studied at quantum level

[Skvortsov, Tran, Tsulaia ’18’20]

Twistor space and free differential algebra reformulations are available

[Krasnov, Skvortsov, Tran ’21; Skvortsov, Van Dongen ’22;
Sharapov, Skvortsov, Sukhanov, Van Dongen ’22]

Chiral theory

No-go theorems. Despite the theory has a non-linear action, the amplitudes are, in effect, trivial. Accordingly, there is no contradiction with the no-go results (e.g. Coleman-Mandula theorem).

Parity-invariant completion. If exists, its scattering is expected to be more non-trivial (no self-duality, hence no integrability and amplitudes are less trivial).

Direct analysis in the light-cone gauge shows that there is no local parity-invariant completion. The same, however, applies to theories in AdS as well.

This is why we attempt here to go beyond the self-dual sector using higher-spin symmetries — at least this works in AdS.

Chiral theory

What we will do: consider 2-pt and 3-pt functions in the chiral theory and try to identify *the associative HS product* and *the cyclic trace*, which will enable us to construct HS invariant higher-point amplitudes

2-point amplitudes

By two-point amplitudes in flat space we understand the Wightman functions. For scalar fields one has

$$G_2^0 = \int d^4p_1 d^4p_2 \theta(p_1^0)\delta(p_1^2)\delta^4(p_1+p_2)\Phi_1(p_1)\Phi_2(p_2).$$

Converting this to the spinor-helicity representation, we obtain

$$A_2^0 = \langle 1\mu\rangle[\mu 1]\delta(\langle 1\mu\rangle[\mu 1] + \langle 2\mu\rangle[\mu 2])\delta(\langle 12\rangle)\delta([12]).$$

Note that it is not manifestly Lorentz covariant due to the presence of the reference spinor.

Analogously, for helicity-h two-point function one finds

$$A_2^h = \left(-\frac{[1\mu]\langle\mu 2\rangle}{[2\mu]\langle\mu 1\rangle}\right)^h \langle 1\mu\rangle[\mu 1]\delta(\langle 1\mu\rangle[\mu 1] + \langle 2\mu\rangle[\mu 2])\delta(\langle 12\rangle)\delta([12]).$$

2-point amplitudes

To bring it to the form, which is reminiscent of that in AdS, we sum it over spins

$$A_2 = \sum_{h=-\infty}^{\infty} \left(-\frac{\langle 1\mu\rangle[\mu 2]}{\langle 2\mu\rangle[\mu 1]} \right)^h \langle 1\mu\rangle[\mu 1]\delta(\langle 1\mu\rangle[\mu 1] + \langle 2\mu\rangle[\mu 2])\delta(\langle 12\rangle)\delta([12]).$$

To perform the sum, we use the following standard regularisation

$$\sum_{h=-\infty}^{\infty} z^h = \delta(1-z).$$

This gives

$$A_2 = \delta\left(\langle 2\mu\rangle[\mu 1] + \langle 1\mu\rangle[\mu 2]\right) \langle 2\mu\rangle[\mu 1]\langle 1\mu\rangle[\mu 1]\delta(\langle 1\mu\rangle[\mu 1] + \langle 2\mu\rangle[\mu 2])\delta(\langle 12\rangle)\delta([12]).$$

By going to new arguments of delta-functions, this can be written as

$$A_2 = \delta^2(\lambda_1 - \lambda_2)\delta^2(\bar{\lambda}_1 + \bar{\lambda}_2).$$

3-point amplitudes

We need to sum

$$A_3^{h_1,h_2,h_3} = g\frac{\ell^{h-1}}{(h-1)!}[12]^{h_1+h_2-h_3}[23]^{h_2+h_3-h_1}[31]^{h_3+h_1-h_2}\delta^4(\lambda_1\bar{\lambda}_1+\lambda_2\bar{\lambda}_2+\lambda_3\bar{\lambda}_3)$$

over helicities on each leg. With the previous regularisation this gives

$$A_3 = g[12]^3 e^{\ell[12]}\delta([12]-[23])\delta([12]-[31])\delta^4(\lambda_1\bar{\lambda}_1+\lambda_2\bar{\lambda}_2+\lambda_3\bar{\lambda}_3).$$

One can further simplify this expression by changing arguments of delta functions

$$A_3 = g e^{\ell[12]}\delta^2(\bar{\lambda}_1+\bar{\lambda}_2+\bar{\lambda}_3)\delta^2(\lambda_2-\lambda_3)\delta^2(\lambda_1-\lambda_3).$$

It is very reminiscent of the result that we have in AdS!

Algebraic structures

Following the AdS setup, we introduce the associative product

$$(\Phi_1 \ltimes \Phi_2)(\lambda_3, \bar{\lambda}_3) \equiv \int d^2\lambda_1 d^2\bar{\lambda}_1 d^2\lambda_2 d^2\bar{\lambda}_2 \Phi_1(\lambda_1, \bar{\lambda}_1)\Phi_2(\lambda_2, \bar{\lambda}_2) e^{\ell[12]} \delta^2(\bar{\lambda}_1 + \bar{\lambda}_2 - \bar{\lambda}_3)\delta^2(\lambda_2 - \lambda_3)\delta^2(\lambda_1 - \lambda_3)$$

and trace, which is cyclic with respect to it

$$\mathrm{tr}_{\ltimes}(\Phi(\lambda, \bar{\lambda})) \equiv \int d^2\lambda d^2\bar{\lambda} \Phi(\lambda, \bar{\lambda})\delta^2(\bar{\lambda}), \qquad \mathrm{tr}_{\ltimes}(\Phi_1 \ltimes \Phi_2) = \mathrm{tr}_{\ltimes}(\Phi_2 \ltimes \Phi_1).$$

These are chosen so that the kernels of

$$G_2 = \mathrm{tr}_{\ltimes}(\Phi_1 \ltimes \Phi_2), \qquad G_3 = \mathrm{tr}_{\ltimes}(\Phi_1 \ltimes \Phi_2 \ltimes \Phi_3)$$

reproduce amplitudes that we have just computed

Higher-spin algebra in flat space

Associativity of the product and cyclicity of the trace implies that A_2 and A_3 are invariant under

$$\bar{\delta}_\varepsilon \Phi \equiv [\Phi, \varepsilon]_{\ltimes} \equiv \Phi \ltimes \varepsilon - \varepsilon \ltimes \Phi.$$

In this way we find that chiral higher-spin theories have some global higher-spin symmetry. This was not built in!

Still, relevance of this algebra was seen before when reformulating the chiral higher-spin theory as the self-dual theory, in terms of twistors and free differential algebras

[DP ’17; Krasnov, Skvortsov, Tran ’21; Skvortsov, Van Dongen ’22; Sharapov, Skvortsov, Sukhanov, Van Dongen ’22]

Higher-point amplitudes

In the same way as in AdS, one can construct higher point amplitudes

$$G_n \equiv \mathrm{tr}_{\ltimes}(\Phi_1 \ltimes \Phi_2 \ltimes \cdots \ltimes \Phi_n),$$

which are manifestly higher-spin invariant.

Properties

Computing explicitly we find

$$G_n = \int \prod_{i=1}^{n} d^2\lambda_i d^2\bar{\lambda}_i \Phi_i(\lambda_i, \bar{\lambda}_i) \prod_{n\geq i>j\geq 2} e^{\ell[ji]} \delta^2(\sum_{i=1}^{n} \bar{\lambda}_i) \prod_{i=2}^{n} \delta^2(\lambda_1 - \lambda_i).$$

For four-point function one gets

$$A_4 = e^{\ell([23]+[24]+[34])} \delta^2(\bar{\lambda}_1 + \bar{\lambda}_2 + \bar{\lambda}_3 + \bar{\lambda}_4) \delta^2(\lambda_1 - \lambda_2) \delta^2(\lambda_1 - \lambda_3) \delta^2(\lambda_1 - \lambda_4).$$

It has interesting features:

1) Scattering occurs at all lambda equal
2) Barred lambda is conserved separately
3) This means that scattering is non-trivial only for p_i p_j =0. That is all Mandelstam variables are vanishing
4) Chiral, relies on complex momenta

Properties

By making the Fourier transform

$$\Phi(\lambda,\bar{\lambda}) = \frac{1}{4\pi^2}\int d^2\bar{\mu} e^{i\bar{\mu}\bar{\lambda}}\Upsilon(\lambda,\bar{\mu}), \qquad \Upsilon(\lambda,\bar{\mu}) \equiv \int d^2\bar{\lambda} e^{i\bar{\lambda}\bar{\mu}}\Phi(\lambda,\bar{\lambda}).$$

The original product goes into

$$(\Upsilon_1\bar{\circ}\Upsilon_2)(\lambda_3,\bar{\mu}_3) \equiv \frac{1}{4\pi^2\ell}\int d^2\lambda_1 d^2\bar{\mu}_1 d^2\lambda_2 d^2\bar{\mu}_2 \Upsilon_1(\lambda_1,\bar{\mu}_1)\Upsilon_2(\lambda_2,\bar{\mu}_2)$$
$$e^{\frac{1}{\ell}([\mu_1\mu_2]+[\mu_2\mu_3]+[\mu_3\mu_1])}\delta^2(\lambda_2-\lambda_3)\delta^2(\lambda_1-\lambda_3).$$

It behaves as the AdS star product in barred mu variable and as a trivial commutative product on lambda variable. Quadratic polynomials

$$P_{\alpha\dot{\alpha}} \sim \lambda_\alpha\bar{\mu}_{\dot{\alpha}}, \qquad \bar{J}_{\dot{\alpha}\dot{\alpha}} \sim \bar{\mu}_{\dot{\alpha}}\bar{\mu}_{\dot{\alpha}}$$

generate part of the Poincare algebra. The remaining J is no part of the chiral higher-spin algebra. (Though, amplitudes still have it as a manifest symmetry)

Properties

One may try to cure chirality of amplitudes by adding

$$G_n \equiv \mathrm{tr}_{\rtimes}(\Phi_1 \rtimes \Phi_2 \rtimes \cdots \rtimes \Phi_n),$$

where

$$(\Phi_1 \rtimes \Phi_2)(\lambda_3, \bar{\lambda}_3) \equiv \int d^2\lambda_1 d^2\bar{\lambda}_1 d^2\lambda_2 d^2\bar{\lambda}_2 \Phi_1(\lambda_1, \bar{\lambda}_1)\Phi_2(\lambda_2, \bar{\lambda}_2) e^{\ell\langle 12\rangle}\delta^2(\lambda_1 + \lambda_2 - \lambda_3)\delta^2(\bar{\lambda}_2 - \bar{\lambda}_3)\delta^2(\bar{\lambda}_1 - \bar{\lambda}_3)$$

is parity conjugate to the original $\ltimes$ product. Unlike in AdS space, however, amplitudes above are not invariant with respect to the original symmetry

$$\bar{\delta}_\varepsilon \Phi \equiv [\Phi, \varepsilon]_{\ltimes} \equiv \Phi \ltimes \varepsilon - \varepsilon \ltimes \Phi.$$

So, the naive way of curing parity by adding parity-conjugate amplitudes, unlike in AdS, breaks the original symmetry of the theory.

Conclusion

Conclusion

1) We regularised the sums over helicities in 2-pt and 3-pt amplitudes of chiral higher-spin theories in flat space
2) The resulting amplitudes quite manifestly have the form of invariant traces of a certain associative algebra. This pattern closely mimics the one in AdS, which was confirmed holographically.
3) This ensures that the chiral higher-spin theory has a certain global higher-spin algebra as a symmetry.
4) Using the associative product and the respective cyclic trace extracted from 2-pt and 3-pt functions, one can construct manifestly higher-spin invariant higher-point amplitudes
5) This gives us *first flat space amplitudes in higher-spin gauge theories, which are non-vanishing beyond 3-point level*
6) Amplitudes involve distributions

Further directions

1) Restoring parity-invariance. Unlike in AdS, naive addition of parity-conjugate amplitudes breaks higher-spin symmetry. So, in the current form, amplitudes are chiral. This means, at least, that these crucially rely on complex momenta
2) What is the theory (action) these amplitudes correspond to? Is it local?
3) Fix undetermined relative factors for each n-point amplitude. This may require developing the holographic description of this theory.

Thank you!

External lines

As usual, on the external lines of the S-matrix one has the on-shell states, which are solutions to the free equations of motion. For massless fields in flat space EOM's in the covariant form read

$$\begin{aligned} \eta_{aa}\varphi^{a(s)} &= 0, \\ \Box\varphi^{a(s)} &= 0, \\ \partial_a\varphi^{a(s)} &= 0 \end{aligned}$$

Gauge transformations are given by

$$\delta\varphi^{a(s)} = \partial^a\xi^{a(s-1)}$$

$$\begin{aligned} \eta_{aa}\xi^{a(s-1)} &= 0, \\ \Box\xi^{a(s-1)} &= 0, \\ \partial_a\xi^{a(s-1)} &= 0 \end{aligned}$$

These are usually solved in the Fourier space.

Constraints from gauge invariance

Solutions from the previous slide define massless representations of the Poincare algebra. Amplitudes are Poincare invariant forms on these representations

$$A_{a_1(s_1),\ldots a_n(s_n)}(p_1,\ldots,p_n)=M_{a_1(s_1),\ldots a_n(s_n)}(p_1,\ldots,p_n)\delta^d(p_1+\cdots+p_n)$$

Gauge invariance leads to the familiar Ward identities in massless theories

$$p_i^{a_i}M_{a_1(s_1),\ldots a_n(s_n)}(p_1,\ldots,p_n)=0,\qquad \forall i.$$

The Ward identities are, however, *approach-dependent*. In particular, one can use instead of phi their gauge-fixed counterparts. Then, there will be no gauge symmetries and no Ward identities. *Global symmetries*, in turn, are more universal

Global symmetries

Global symmetries in gauge theories occur as follows. One should look into the kernel of the free gauge transformation

$$\delta\varphi^{a(s)} = \partial^a \tilde{\xi}^{a(s-1)} = 0.$$

Parameters that solve eqn above generate global symmetry transformations. In the non-linear theory this happens as follows

$$\delta^{nl}_{\tilde{\xi}} \varphi^{a(s)} = \partial^a \tilde{\xi}^{a(s-1)} + T(\tilde{\xi}, \varphi) + \dots$$

where T is linear in phi and xi and gives the first non-linear correction to the gauge transformation law. Global symmetries are generated by

$$\delta^{gl}_{\tilde{\xi}} \varphi^{a(s)} = T(\tilde{\xi}, \varphi).$$

They still survive in a gauge-fixed theory.

Examples

The Yang-Mills theory. Gauge transformations in the free theory are

$$\delta A^a(x) = \partial^a \xi(x).$$

So, the global symmetry parameters are x-independent. In the non-linear theory they generate

$$\delta_{\tilde{\xi}} A^a(x) = \partial^a \tilde{\xi} + [A(x), \tilde{\xi}] = [A(x), \tilde{\xi}]$$

which are, indeed, the global transformations in internal space.

Examples

Gravity. Gauge transformations in the free theory are

$$\delta g^{aa}(x) = \partial^a \xi^a(x).$$

Global parameters are just the Killing vectors

$$\tilde{\xi}^a(x) = a^a + \omega^{a,b} x_b, \qquad \omega_{a,b} = -\omega_{b,a}.$$

In the non-linear theory, these generate the flat space isometries, that is the global Poincare algebra

$$\delta_{\tilde{\xi}} g^{aa}(x) = \mathcal{L}_{\tilde{\xi}} g^{aa}(x).$$

Higher-spin case

In the *general spin case* global symmetry parameters

$$\partial^a \tilde{\xi}^{a(s-1)} = 0$$

are given by the traceless Killing tensors of the Minkowski space.

This defines the spectrum of the global higher-spin algebra.

Further consistency conditions

Global symmetry transformations should close into themselves

$$[\delta_{\tilde{\xi}_1}, \delta_{\tilde{\xi}_2}]\varphi = \delta_{\tilde{\xi}_3}\varphi \equiv \delta_{[\tilde{\xi}_1,\tilde{\xi}_2]}\varphi,$$

which defines the commutator of global symmetries. It should satisfy the Jacobi identity, that is global symmetries form a Lie algebra. If we want to have gravity as spin-2, it should have the Poincare subalgebra

Finally,

$$\delta_{\tilde{\xi}}\varphi \to \varphi$$

should be a representation of this algebra. Moreover, under the Poincare subalgebra, fields should transform in the massless higher-spin representations that we started from.